Heterotic standard model moduli

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ABSTRACT: In previous papers, we introduced a heterotic standard model and discussed its basic properties. The Calabi-Yau threefold has, generically, three Kähler and three complex structure moduli. The observable sector of this vacuum has the spectrum of the MSSM with one additional pair of Higgs-Higgs conjugate fields. The hidden sector has no charged matter in the strongly coupled string and only minimal matter for weak coupling. Additionally, the spectrum of both sectors will contain vector bundle moduli. The exact number of such moduli was conjectured to be small, but was not explicitly computed. In this paper, we rectify this and present a formalism for computing the number of vector bundle moduli. Using this formalism, the number of moduli in both the observable and strongly coupled hidden sectors is explicitly calculated.

Contents

1. Introduction

There is a long history in the search for realistic compactifications of the heterotic string, see [1 – 19]. But until recently, finding compactifications yielding a viable particle spectrum had resisted all efforts. In a series of papers [20 – 22], we presented a "heterotic standard model" of particle physics. Specifically, we presented a small class of $E_8 \times E_8$ heterotic superstring vacua whose observable sectors have the spectrum of the minimal supersymmetric standard model (MSSM), with the exception of one additional pair of Higgs-Higgs conjugate superfields, and *no exotic multiplets*. Such vacua occur for both weak and strong string coupling.

Technically, our heterotic standard vacua consist of stable, holomorphic vector bundles, V, with structure group SU(4) over elliptically fibered Calabi-Yau threefolds, X, with a $\mathbb{Z}_3 \times \mathbb{Z}_3$ fundamental group. These bundles admit a gauge connection that, in conjunction with a Wilson line, spontaneously breaks the observable sector E_8 gauge group down to the $SU(3)_C \times SU(2)_L \times U(1)_Y$ standard model group times an additional $U(1)_{B-L}$ symmetry. The spectrum arises as the $\mathbb{Z}_3 \times \mathbb{Z}_3$ invariant sheaf cohomology.

The existence of elliptically fibered Calabi-Yau threefolds with $\mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ fundamental group was first demonstrated in [23–25] and [26, 27] respectively. More recently, elliptic Calabi-Yau threefolds with $\mathbb{Z}_3 \times \mathbb{Z}_3$ fundamental group were constructed and classified [28]. Methods for building stable, holomorphic vector bundles with arbitrary structure group in E_8 over simply connected elliptic Calabi-Yau threefolds were introduced in [29–32] and greatly expanded in a number of papers [23–25, 33–35]. These constructions were then generalized to elliptically fibered Calabi-Yau threefolds with non-trivial fundamental group in [25–27, 36]. In order to obtain a realistic spectrum, it was found necessary to introduce a new method [23–27] for constructing vector bundles. This consists of building the requisite bundles by "extension" from simpler, lower rank bundles. This method was used for manifolds with $\mathbb{Z}_2$ fundamental group in [25, 37, 38] and in the heterotic standard model context in [28]. In recent work [20–22, 37, 38], it was shown how to compute the complete low-energy spectrum of such vacua. This requires one to evaluate the relevant sheaf cohomologies, find the action of the finite fundamental group on these spaces and, finally, to tensor this with the action of the Wilson line on the associated representation. The low energy spectrum is the invariant cohomology subspaces under the resulting group action. This was applied in [20–22] to compute the exact spectrum of all multiplets transforming *non-trivially* under the action of the low energy gauge group. The accompanying natural method of "doublet-triplet" splitting was also discussed.

Although a similar calculation in principle, the spectrum of gauge *singlet* superfields was only partially determined. In addition to the three Kähler and three complex structure moduli, there are vector bundle moduli whose number was not computed. The reason is that the relevant cohomology space lies in a complex of intertwined long exact sequences which makes it, in general, much harder to evaluate than the other sheaf cohomologies. Be this as it may, vector bundle moduli are important in the particle phenomenology of these vacua, contributing, for example, to the mu-terms and Yukawa couplings. Furthermore, these moduli are central to the discussion of vacuum stability [39–48], the cosmological constant [49–51], and cosmology [52–54]. Hence, it is essential that their spectrum be computed.

In this paper, we present a general formalism for evaluating the number of gauge singlet superfields for vector bundles constructed by extension. We then apply this method to explicitly compute the number of vector bundle moduli in both the observable and hidden sectors of a heterotic standard model. Specifically, we do the following. In section 2, all relevant properties of both the Calabi-Yau threefold and the holomorphic vector bundles of the heterotic standard vacua are outlined. For clarity, we present our discussion and formalism in terms of the observable sector vector bundle. The hidden sector bundle will be introduced in the final section. The sheaf cohomologies and their relation to the low-energy spectrum are briefly discussed and the cohomology space of vector bundle moduli is presented. The relevant short and long exact sequences are given in section 3. The various cohomologies in the intertwined complex of long exact sequences are systematically calculated using two Leray spectral sequences. In section 5 all this information is brought together to compute the number of vector bundle moduli. For the heterotic standard model vacua under consideration, the number of such moduli in the observable sector is found to

Moduli	Kähler	Complex structure	Vector Bundle (visible E_8)	Vector Bundle (hidden E_8)
Number	3	3	19	5

Table 1: Moduli fields in "A Heterotic Standard Model".

be $n_{\text{observable}} = 19$. Finally, this formalism is applied in section 6 to compute the number of vector bundle moduli in the strongly coupled hidden sector. We find that $n_{hidden} = 5$. To summarize, the moduli fields are listed in table 1.

2. Preliminaries

In our approach, there are two fundamental ingredients needed to construct a heterotic standard model. The first is a class of Calabi-Yau threefolds X with fundamental group $\mathbb{Z}_3 \times \mathbb{Z}_3$. The second consists of (a moduli space of) stable, holomorphic vector bundles V over X with structure group SU(4) which satisfy appropriate physical constraints. Calabi-Yau threefolds of this type were constructed in [28]. Similarly, in [22] the requisite holomorphic vector bundles were discussed in detail. Here, we simply outline the properties of X and V that are relevant to this paper.

2.1 The Calabi-Yau threefold X

The Calabi-Yau threefold, X, is constructed as follows. Begin by considering a simply connected Calabi-Yau threefold, $\widetilde{X}$, which is an elliptic fibration over a $d\mathbb{P}_9$ surface. It was shown in [28] that there are special $d\mathbb{P}_9$ surfaces which admit a $\mathbb{Z}_3 \times \mathbb{Z}_3$ action. A suitable fiber product of two such $d\mathbb{P}_9$ surfaces is then a Calabi-Yau threefold with an induced fixed-point free $\mathbb{Z}_3 \times \mathbb{Z}_3$ group action. Hence, the quotient $X = \widetilde{X}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is a smooth Calabi-Yau threefold that is torus-fibered over a singular $d\mathbb{P}_9$ and has non-trivial fundamental group $\mathbb{Z}_3 \times \mathbb{Z}_3$, as desired.

Specifically, $\widetilde{X}$ is a fiber product

$$\widetilde{X} = B_1 \times_{\mathbb{P}^1} B_2 \tag{2.1}$$

of two special $d\mathbb{P}_9$ surfaces B_1 and B_2. Thus, $\widetilde{X}$ is elliptically fibered over both surfaces with the projections

$$\pi_1 : \widetilde{X} \to B_1, \qquad \pi_2 : \widetilde{X} \to B_2. \tag{2.2}$$

The surfaces B_1 and B_2 are themselves elliptically fibered over $\mathbb{P}^1$ with maps

$$\beta_1 : B_1 \to \mathbb{P}^1, \qquad \beta_2 : B_2 \to \mathbb{P}^1. \tag{2.3}$$

Together, these projections yield the commutative diagram

$$
\begin{array}{cccc}
\dim_{\mathbb{C}} = 3 : & & \widetilde{X} & \\
& {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \pi_2} \\
\dim_{\mathbb{C}} = 2 : & B_1 & & B_2 \\
& {\scriptstyle \beta_1} \searrow & & \swarrow {\scriptstyle \beta_2} \\
\dim_{\mathbb{C}} = 1 : & & \mathbb{P}^1 &.
\end{array}
\tag{2.4}
$$

The invariant homology ring of each special $d\mathbb{P}_9$ surface is generated by two $\mathbb{Z}_3 \times \mathbb{Z}_3$ invariant curve classes f and t with intersections

$$f^2 = 0, \qquad ft = 3t^2 = 3. \tag{2.5}$$

Using projections (2.2), these can be lifted to divisor classes

$$\tau_1 = \pi_1^{-1}(t_1), \qquad \tau_2 = \pi_2^{-1}(t_2), \qquad \phi = \pi_1^{-1}(f_1) = \pi_2^{-1}(f_2) \tag{2.6}$$

on $\widetilde{X}$ satisfying the intersection relations

$$\phi^2 = \tau_1^3 = \tau_2^3 = 0, \qquad \phi\tau_1 = 3\tau_1^2, \qquad \phi\tau_2 = 3\tau_2^2. \tag{2.7}$$

These three classes generate the invariant homology ring of $\widetilde{X}$. For example, one can show that X has, generically, six geometric moduli; three Kähler moduli and three complex structure moduli.

Finally, the Chern classes of $\widetilde{X}$ are found to be

$$c_1\big(T\widetilde{X}\big) = c_3\big(T\widetilde{X}\big) = 0, \qquad c_2\big(T\widetilde{X}\big) = 12(\tau_1^2 + \tau_2^2). \tag{2.8}$$

2.2 The observable sector bundle V

The observable sector bundles V on X are produced by constructing stable, holomorphic vector bundles $\widetilde{V}$ with structure group $SU(4) \subset E_8$ over $\widetilde{X}$ that are equivariant under the action of $\mathbb{Z}_3 \times \mathbb{Z}_3$. Then $V = \widetilde{V}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$. One further requires that V and, hence, $\widetilde{V}$ satisfy the appropriate physical constraints.

The vector bundles $\widetilde{V}$ are constructed using a generalization of the method of "bundle extensions" [25, 27]. Specifically, $\widetilde{V}$ is the extension[1]

$$0 \longrightarrow V_2 \longrightarrow \widetilde{V} \longrightarrow V_1 \longrightarrow 0 \tag{2.9}$$

of two rank two bundles V_1 and V_2 on $\widetilde{X}$. These are of the form

$$V_i = \mathcal{L}_i \otimes \pi_2^* W_i, \qquad i = 1, 2 \tag{2.10}$$

for some line bundles $\mathcal{L}_i$ on $\widetilde{X}$ and rank 2 bundles W_i on B_2. The rank two bundles W_i are themselves extensions

$$0 \longrightarrow \mathcal{O}_{B_2}(a_i f_2) \longrightarrow W_i \longrightarrow \mathcal{O}_{B_2}(b_i f_2) \otimes I_{k_i} \longrightarrow 0, \tag{2.11}$$

where a_i, b_i are integers and I_{k_i} is the ideal sheaf of some k_i-tuple of points on B_2.

One must specify not only the bundles $\widetilde{V}$, but their transformations under $\mathbb{Z}_3 \times \mathbb{Z}_3$ as well. To do this, first notice that for the $\mathbb{Z}_3 \times \mathbb{Z}_3$ action on the space of extensions to be well-defined, the line bundles $\mathcal{O}_{B_2}(a_i f_2)$, $\mathcal{O}_{B_2}(b_i f_2)$ and $\mathcal{L}_i$ must be equivariant under the finite group action. In this case, the space of extensions will carry a representation of

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[1]The attentive reader will notice that we exchanged V_1 and V_2 in the sequence as compared to [20–22]. This just means that we are working at a slightly different point in the Kähler and vector bundle moduli space. As one can easily check, the particle spectrum is unchanged.

$\mathbb{Z}_3 \times \mathbb{Z}_3$. An invariant class in the extension space defines an equivariant vector bundle extension. A rank 4 vector bundle $\widetilde{V}$ with this property will inherit an explicit equivariant structure from the action of $\mathbb{Z}_3 \times \mathbb{Z}_3$ on its constituent line bundles. Having found such a $\widetilde{V}$, one can construct $V = \widetilde{V}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ on X.

As discussed in [20 – 22], the requirement that V admit a gauge connection which satisfies the hermitian Yang-Mills equations and leads to three chiral families of quarks/leptons, no exotic matter and two pairs of Higgs-Higgs conjugate fields (the minimal number) imposes strong constraints on $\widetilde{V}$. These are the following. First, in order for the hermitean Yang-Mills gauge connection to exist on $\widetilde{V}$ this vector bundle must be (slope) stable. A non-trivial set of necessary conditions for stability are

$$H^0\left(\widetilde{X}, \widetilde{V}\right) = H^0\left(\widetilde{X}, \widetilde{V}^\vee\right) = 0, \qquad H^0\left(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\right) = 1. \tag{2.12}$$

The remaining three physical constraints were shown in [22] to require that

$$c_3\left(\widetilde{V}\right) = -54, \qquad h^1\left(\widetilde{X}, \widetilde{V}^\vee\right) = 0, \qquad h^1\left(\widetilde{X}, \wedge^2\widetilde{V}\right) = 14 \tag{2.13}$$

respectively.

A unique (up to continuous moduli) solution for $\widetilde{V}$ that is compatible with all of our constraints[2] was found in [22]. It is constructed as follows. First consider the rank two bundles W_i for $i = 1, 2$ on B_2. Take W_1 to be

$$W_1 = \mathcal{O}_{B_2} \oplus \mathcal{O}_{B_2}. \tag{2.14}$$

Note that this is the trivial extension of (2.11) with $a_1 = b_1 = k_1 = 0$. Now let W_2 be an equivariant bundle in the space of extension of the form

$$0 \longrightarrow \mathcal{O}_{B_2}(-2f_2) \longrightarrow W_2 \longrightarrow \chi_2 \mathcal{O}_{B_2}(2f_2) \otimes I_9 \longrightarrow 0, \tag{2.15}$$

where for the ideal sheaf I_9 of 9 points we take a generic $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbit. Second, choose the two line bundles $\mathcal{L}_i$ for $i = 1, 2$ on $\widetilde{X}$ to be

$$\mathcal{L}_1 = \chi_2 \mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2) \tag{2.16}$$

and

$$\mathcal{L}_2 = \mathcal{O}_{\widetilde{X}}(\tau_1 - \tau_2) \tag{2.17}$$

respectively. Here, χ_1 and χ_2 are the two natural one-dimensional representations of $\mathbb{Z}_3 \times \mathbb{Z}_3$ defined by

$$\chi_1(g_1) = \omega, \qquad \chi_1(g_2) = 1; \qquad \chi_2(g_1) = 1, \qquad \chi_2(g_2) = \omega, \tag{2.18}$$

where $g_{1,2}$ are the generators of the two $\mathbb{Z}_3$ factors, $\chi_{1,2}$ are two group characters of $\mathbb{Z}_3 \times \mathbb{Z}_3$, and $\omega = e^{\frac{2\pi i}{3}}$ is a third root of unity.

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[2]We verified that the cohomology groups of $\widetilde{V}$ satisfy the constraints eq. (2.12) imposed by stability. We also have not been able to find any destabilizing subbundle, and it appears to be stable to experts in the field. Stability was proven for a very similar vector bundle in [27], and we expect the same methods to work in our case. Of course, ultimately one has to give a detailed mathematical proof. This will be presented elsewhere.

It follows that the two rank 2 bundles $V_{1,2}$ defined in eq. (2.10) are given by

$$V_1 = \chi_2 \mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2) \oplus \chi_2 \mathcal{O}_{\widetilde{X}}(-\tau_1 + \tau_2)$$
$$V_2 = \mathcal{O}_{\widetilde{X}}(\tau_1 - \tau_2) \otimes \pi_2^* W_2\,. \tag{2.19}$$

The observable sector bundle $\widetilde{V}$ is then an equivariant element of the space of extensions eq. (2.9).

2.3 Computing the particle spectrum

As discussed in detail in [22], the low-energy particle spectrum is given by

$$\ker(\slashed{D}_{\widetilde{V}}) = \left(H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \otimes \mathbf{45}\right)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \oplus \left(H^1(\widetilde{X}, \mathrm{ad}(\widetilde{V})) \otimes \mathbf{1}\right)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \oplus$$
$$\oplus \left(H^1(\widetilde{X}, \widetilde{V}) \otimes \mathbf{16}\right)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \oplus \left(H^1(\widetilde{X}, \widetilde{V}^\vee) \otimes \overline{\mathbf{16}}\right)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \oplus$$
$$\oplus \left(H^1(\widetilde{X}, \wedge^2 \widetilde{V}) \otimes \mathbf{10}\right)^{\mathbb{Z}_3 \times \mathbb{Z}_3},$$

where the superscript indicates the invariant subspace under the action of $\mathbb{Z}_3 \times \mathbb{Z}_3$. The invariant cohomology space $(H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \otimes \mathbf{45})^{\mathbb{Z}_3 \times \mathbb{Z}_3}$ corresponds to gauge superfields in the low-energy spectrum carrying the adjoint representation of $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \times \mathrm{U}(1)_{B-L}$. The matter cohomology spaces, $(H^1(\widetilde{X}, \widetilde{V}) \otimes \mathbf{16})^{\mathbb{Z}_3 \times \mathbb{Z}_3}$, $(H^1(\widetilde{X}, \widetilde{V}^\vee) \otimes \overline{\mathbf{16}})^{\mathbb{Z}_3 \times \mathbb{Z}_3}$ and $(H^1(\widetilde{X}, \wedge^2 \widetilde{V}) \otimes \mathbf{10})^{\mathbb{Z}_3 \times \mathbb{Z}_3}$ were all explicitly computed in [22], leading to three chiral families of quarks/leptons (each family with a right-handed neutrino [55]), no exotic superfields and two vector-like pairs of Higgs-Higgs conjugate superfields respectively. The remaining cohomology space in eq. (2.20), namely,

$$\left(H^1(\widetilde{X}, \mathrm{ad}(\widetilde{V})) \otimes \mathbf{1}\right)^{\mathbb{Z}_3 \times \mathbb{Z}_3}, \tag{2.20}$$

corresponds to the vector bundle moduli in the low-energy spectrum, see also [8, 56–59, 48, 47]. Since $\mathrm{ad}(\widetilde{V})$ is a rank 15 vector bundle, its cohomology is much harder to compute than the previous cohomology spaces and, for that reason, was not evaluated in [20–22]. However, vector bundle moduli play an essential role in mu-terms, Yukawa couplings and in the discussion of vacuum stability and the cosmological constant. For these reasons, and to complete the spectrum, this paper will present a formalism for computing eq. (2.20). We will then use this formalism to explicitly evaluate the number of vector bundle moduli in the heterotic standard model.

3. The exact sequences

It is clear from eq. (2.20) that we must compute the cohomology space $H^1(\widetilde{X}, \mathrm{ad}(\widetilde{V}))$. First, recall that the action of the Wilson line on the $\mathbf{1}$ representation is trivial. Hence, we only need to know the $\mathbb{Z}_3 \times \mathbb{Z}_3$-invariant part of the cohomology. Second, note that $\mathrm{ad}(\widetilde{V})$ is defined to be the traceless part of $\widetilde{V} \otimes \widetilde{V}^\vee$. But the trace part is just the trivial line bundle, whose first cohomology group vanishes. It follows that the vector bundle moduli

are precisely

$$H^1\left(\widetilde{X}, \mathrm{ad}(\widetilde{V})\right)^{\mathbb{Z}_3 \times \mathbb{Z}_3} = H^1\left(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\right)^{\mathbb{Z}_3 \times \mathbb{Z}_3} - \underbrace{H^1\left(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\right)^{\mathbb{Z}_3 \times \mathbb{Z}_3}}_{=0}. \qquad (3.1)$$

Therefore, the tangent space to the moduli space is $H^1\left(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\right)^{\mathbb{Z}_3 \times \mathbb{Z}_3}$. To compute this space, one must consider complexes of interlocking exact sequences.

3.1 Short exact bundle sequences

Recall from eq. (2.9) that the vector bundle $\widetilde{V}$ is defined by the short exact sequence of bundles

$$0 \longrightarrow V_2 \longrightarrow \widetilde{V} \longrightarrow V_1 \longrightarrow 0. \qquad (3.2)$$

One can tensor this sequence on the right by the bundles $V_1^\vee$, $\widetilde{V}^\vee$ and $V_2^\vee$ to produce three new short exact sequences which we will refer to as (a), (b) and (c) respectively. Now take the dual of eq. (3.2). This gives the short exact sequence of bundles

$$0 \longrightarrow V_1^\vee \longrightarrow \widetilde{V}^\vee \longrightarrow V_2^\vee \longrightarrow 0, \qquad (3.3)$$

which we will tensor with vector bundles V_2, $\widetilde{V}$ and V_1. Of course, the tensor product is commutative, but we will write it as tensoring on the left. The three resulting short exact sequences will be referred to as (d), (e) and (f) respectively. The six short exact bundle sequences constructed in this manner can be written together as the commutative diagram of exact sequences

$$\begin{array}{ccccccc}
 & (d) & & (e) & & (f) & \\
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
(a) \quad 0 \longrightarrow & V_2 \otimes V_1^\vee & \longrightarrow & \widetilde{V} \otimes V_1^\vee & \longrightarrow & V_1 \otimes V_1^\vee & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
(b) \quad 0 \longrightarrow & V_2 \otimes \widetilde{V}^\vee & \longrightarrow & \widetilde{V} \otimes \widetilde{V}^\vee & \longrightarrow & V_1 \otimes \widetilde{V}^\vee & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
(c) \quad 0 \longrightarrow & V_2 \otimes V_2^\vee & \longrightarrow & \widetilde{V} \otimes V_2^\vee & \longrightarrow & V_1 \otimes V_2^\vee & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array} \qquad (3.4)$$

3.2 Long exact cohomology sequences

Each of the six short exact bundle sequences defined above gives rise to a long exact cohomology sequence. These can also be fit together into the complex of intertwining

sequences of the form[3]

$$
\begin{array}{ccccccccccc}
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to H^{i-2}(V_1 \otimes V_2^\vee) \to & H^{i-1}(V_2 \otimes V_2^\vee) \to & H^{i-1}(\widetilde{V} \otimes V_2^\vee) \to & H^{i-1}(V_1 \otimes V_2^\vee) \to & H^i(V_2 \otimes V_2^\vee) \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to H^{i-1}(V_1 \otimes V_1^\vee) \to & H^i(V_2 \otimes V_1^\vee) \to & H^i(\widetilde{V} \otimes V_1^\vee) \to & H^i(V_1 \otimes V_1^\vee) \to & H^{i+1}(V_2 \otimes V_1^\vee) \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to H^{i-1}(V_1 \otimes \widetilde{V}^\vee) \to & H^i(V_2 \otimes \widetilde{V}^\vee) \to & H^i(\widetilde{V} \otimes \widetilde{V}^\vee) \to & H^i(V_1 \otimes \widetilde{V}^\vee) \to & H^{i+1}(V_2 \otimes \widetilde{V}^\vee) \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to H^{i-1}(V_1 \otimes V_2^\vee) \to & H^i(V_2 \otimes V_2^\vee) \to & H^i(\widetilde{V} \otimes V_2^\vee) \to & H^i(V_1 \otimes V_2^\vee) \to & H^{i+1}(V_2 \otimes V_2^\vee) \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to H^i(V_1 \otimes V_1^\vee) \to & H^{i+1}(V_2 \otimes V_1^\vee) \to & H^{i+1}(\widetilde{V} \otimes V_1^\vee) \to & H^{i+1}(V_1 \otimes V_1^\vee) \to & H^{i+2}(V_2 \otimes V_1^\vee) \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots & \\
\end{array}
$$
(3.5)

where the cohomology spaces in degrees $i < 0$ or $i > 3$ vanish for dimension reasons. Note that the object of interest, namely, $H^1(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee)$, occurs in this complex. By evaluating various other cohomology spaces in these sequences, we will be able to explicitly compute $H^1(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee)$.

3.3 The "corner" cohomologies

We begin by noting that the complex is composed of a number of 3×3 blocks, each of the form

$$
\begin{array}{ccc}
C = H^i\left(V_2 \otimes V_1^\vee\right) & \longmapsto H^i\left(\widetilde{V} \otimes V_1^\vee\right) \longrightarrow & H^i\left(V_1 \otimes V_1^\vee\right) = A \\
\downarrow & \downarrow & \downarrow \\
H^i\left(V_2 \otimes \widetilde{V}^\vee\right) & \longrightarrow H^i\left(\widetilde{V} \otimes \widetilde{V}^\vee\right) \longrightarrow & H^i\left(V_1 \otimes \widetilde{V}^\vee\right) \\
\downarrow & \downarrow & \downarrow \\
D = H^i\left(V_2 \otimes V_2^\vee\right) & \longmapsto H^i\left(\widetilde{V} \otimes V_2^\vee\right) \longrightarrow & H^i\left(V_1 \otimes V_2^\vee\right) = B
\end{array}
\quad ,
$$
(3.6)

containing exclusively degree i cohomology spaces. The cohomology spaces at the corners of each block, labeled as A, B, C and D, are particularly amenable to evaluation, so we begin by computing them.

Cohomologies A. First consider the cohomology spaces

$$
A = H^*\left(\widetilde{X}, V_1 \otimes V_1^\vee\right). \tag{3.7}
$$

It follows from eq. (2.19) that $V_1 \otimes V_1^\vee$ is just the rank 4 trivial bundle,

$$
V_1 \otimes V_1^\vee = \mathcal{O}_{\widetilde{X}}^{\oplus 4}. \tag{3.8}
$$

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[3]To save space, we occasionally suppress the $\widetilde{X}$ in large commutative diagrams.

Then, its cohomology spaces are

$$H^*\Big(\widetilde{X}, V_1 \otimes V_1^\vee\Big) = H^*(\widetilde{X}, \mathcal{O}_{\widetilde{X}})^{\oplus 4} \tag{3.9}$$

and, therefore,

$$\begin{aligned}
H^0\big(\widetilde{X}, V_1 \otimes V_1^\vee\big) &= 4\,, & H^1\big(\widetilde{X}, V_1 \otimes V_1^\vee\big) &= 0\,, \\
H^2\big(\widetilde{X}, V_1 \otimes V_1^\vee\big) &= 0\,, & H^3\big(\widetilde{X}, V_1 \otimes V_1^\vee\big) &= 4\,,
\end{aligned} \tag{3.10}$$

where we have used the simplifying notation that $\mathbb{C}^{\oplus 4} \equiv 1^{\oplus 4} \equiv 4$, thought of as a $\mathbb{Z}_3 \times \mathbb{Z}_3$ representation. In fact, throughout this paper we will often denote the trivial n-dimensional representation by

$$1^{\oplus n} \equiv n\,, \tag{3.11}$$

for any positive integer n.

Cohomologies B. Next, we calculate the spaces B given by

$$H^*\Big(\widetilde{X}, V_1 \otimes V_2^\vee\Big)\,. \tag{3.12}$$

For notational simplicity, we define

$$\mathcal{F} = V_1 \otimes V_2^\vee\,. \tag{3.13}$$

These cohomology spaces are much harder to compute and will be evaluated using several applications of the Leray spectral sequence. The first Leray sequence is associated with integrating over the elliptic fiber of $\pi_2 : \widetilde{X} \to B_2$, hence pushing the cohomology down onto the base surface B_2. In this case, one finds[4]

$$H^i\Big(\widetilde{X}, \mathcal{F}\Big) = \bigoplus_{p,q}^{p+q=i} H^p\Big(B_2, R^q \pi_{2*} \mathcal{F}\Big)\,, \tag{3.14}$$

where the only nonvanishing entries are for $p = 0, 1, 2$ (since $\dim_\mathbb{C} B_2 = 2$) and $q = 0, 1$ (since the fiber of $\widetilde{X}$ is an elliptic curve). It follows from eq. (2.19) that

$$\mathcal{F} = \mathcal{O}_{\widetilde{X}}(-2\tau_1 + 2\tau_2)^{\oplus 2} \otimes \pi_2^* W_2\,, \tag{3.15}$$

where we have used the fact, proven in [22], that $W_2^\vee = \chi_2^2 W_2$. Furthermore, we see from eq. (2.6) that

$$\mathcal{O}_{\widetilde{X}}(\tau_i) = \pi_i^* \mathcal{O}_{B_i}(t_i)\,, \qquad i = 1, 2\,. \tag{3.16}$$

Combining this with eq. (3.15) implies

$$\mathcal{F} = \Big[\pi_1^*\Big(\mathcal{O}_{B_1}(-2t_1)\Big) \otimes \pi_2^*\Big(\mathcal{O}_{B_2}(2t_2) \otimes W_2\Big)\Big]^{\oplus 2}\,. \tag{3.17}$$

Then, using the projection formula and the fact that

$$\big(R^q \pi_{2*}\big) \circ \pi_1^* = \beta_2^* \circ \big(R^q \beta_{1*}\big)\,, \tag{3.18}$$

---

[4]In all the spectral sequences which we are considering higher differentials vanish trivially. Furthermore, there are no extension ambiguities for $\mathbb{C}$-vector spaces.

which follows from the commutativity of the diagram eq. (2.4), one finds

$$R^q \pi_{2*} \mathcal{F} = \left[\beta_2^* R^q \beta_{1*} \left(\mathcal{O}_{B_1}(-2t_1) \right) \otimes \mathcal{O}_{B_2}(2t_2) \otimes W_2 \right]^{\oplus 2} . \qquad (3.19)$$

Using this expression, we can calculate each cohomology space $H^p(B_2, R^q \pi_{2*} \mathcal{F})$ in eq. (3.14), to which we now proceed.

Note that the cohomologies $H^p(B_2, R^q \pi_{2*} \mathcal{F})$ fill out the 2×3 tableau[5]

$$
\begin{array}{c|c|c|c|}
q=1 & H^0\big(B_2, R^1\pi_{2*}\mathcal{F}\big) & H^1\big(B_2, R^1\pi_{2*}\mathcal{F}\big) & H^2\big(B_2, R^1\pi_{2*}\mathcal{F}\big) \\
\cline{2-4}
q=0 & H^0\big(B_2, \pi_{2*}\mathcal{F}\big) & H^1\big(B_2, \pi_{2*}\mathcal{F}\big) & H^2\big(B_2, \pi_{2*}\mathcal{F}\big) \\
\cline{2-4}
& p=0 & p=1 & p=2
\end{array}
\qquad (3.20)
$$

Such tableaux are very useful in keeping track of the elements of Leray spectral sequences. As is clear from eq. (3.14), the sum over the diagonals yields the desired cohomology of $\mathcal{F}$. Let us first evaluate the cohomologies with $q = 0$. Since the curve $-2t_1$ intersects the fiber of B_1 negatively, that is, $-2t_1$ has negative degree, it follows that

$$R^0 \beta_{1*} \left(\mathcal{O}_{B_1}(-2t_1) \right) = \beta_{1*} \left(\mathcal{O}_{B_1}(-2t_1) \right) = 0 . \qquad (3.21)$$

Since the push-down vanishes we immediately obtain

$$H^p \left(B_2, \pi_{2*} \mathcal{F} \right) = 0 , \qquad p = 0, 1, 2 \qquad (3.22)$$

and the Leray tableau eq. (3.20) becomes

$$
\begin{array}{c|c|c|c|}
q=1 & H^0\big(B_2, R^1\pi_{2*}\mathcal{F}\big) & H^1\big(B_2, R^1\pi_{2*}\mathcal{F}\big) & H^2\big(B_2, R^1\pi_{2*}\mathcal{F}\big) \\
\cline{2-4}
q=0 & 0 & 0 & 0 \\
\cline{2-4}
& p=0 & p=1 & p=2
\end{array}
\qquad (3.23)
$$

One must now compute the three cohomologies in the upper row, corresponding to $q = 1$. We begin by using the fact that

$$R^1 \beta_{1*} \mathcal{O}_{B_1}(-2t_1) = \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 6} , \qquad (3.24)$$

derived in [22]. It follows from this and eq. (3.19) that

$$R^1 \pi_{2*} \mathcal{F} = \left[\beta_2^* \left(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 6} \right) \otimes \mathcal{O}_{B_2}(2t_2) \otimes W_2 \right]^{\oplus 2} = \qquad (3.25)$$

$$= \left(\mathcal{O}_{B_2}(2t_2 - f) \otimes W_2 \right)^{\oplus 12} . \qquad (3.26)$$

Using this result, we can now compute $H^p\big(B_2, R^1\pi_{2*}\mathcal{F}\big)$ by pushing down onto the base $\mathbb{P}^1$ of B_2 using a second Leray spectral sequence. This is given for each $p = 0, 1, 2$ by

$$H^p \left(B_2, R^1 \pi_{2*} \mathcal{F} \right) = \bigoplus_{s,t}^{s+t=p} H^s \left(\mathbb{P}^1, R^t \beta_{2*}(R^1 \pi_{2*} \mathcal{F}) \right) , \qquad (3.27)$$

where $s = 0, 1$ (since $\dim_{\mathbb{C}} \mathbb{P}^1 = 1$) and $t = 0, 1$ (since the fiber of B_2 is one dimensional).

---

[5]Of course, the zero-th derived push-down is just the ordinary push-down, $R^0\pi_{2*} = \pi_{2*}$.

From eq. (3.26) and the projection formula, we find that

$$R^t \beta_{2*}(R^1 \pi_{2*} \mathcal{F}) = \left[\mathcal{O}_{\mathbb{P}^1}(-1) \otimes R^t \beta_{2*} \left(\mathcal{O}_{B_2}(2t_2) \otimes W_2 \right) \right]^{\oplus 12}. \tag{3.28}$$

Using this expression, one can calculate the cohomology spaces $H^s \left(\mathbb{P}^1, R^t \beta_{2*}(R^1 \pi_{2*} \mathcal{F}) \right)$ in eq. (3.27).

First note that the cohomologies $H^s \left(\mathbb{P}^1, R^t \beta_{2*}(R^1 \pi_{2*} \mathcal{F}) \right)$ are determined by the 2×2 Leray tableau

$$
\begin{array}{c|c|c|}
t=1 & H^0 \left(\mathbb{P}^1, R^1 \beta_{2*}(R^1 \pi_{2*} \mathcal{F}) \right) & H^1 \left(\mathbb{P}^1, R^1 \beta_{2*}(R^1 \pi_{2*} \mathcal{F}) \right) \\
\hline
t=0 & H^0 \left(\mathbb{P}^1, \beta_{2*}(R^1 \pi_{2*} \mathcal{F}) \right) & H^0 \left(\mathbb{P}^1, \beta_{2*}(R^1 \pi_{2*} \mathcal{F}) \right) \\
\hline
& s=0 & s=1
\end{array}
\tag{3.29}
$$

Let us first evaluate the cohomologies with $t = 1$. Since $2t_2$ has positive degree, it follows that

$$R^1 \beta_{2*} \left(\mathcal{O}_{B_2}(2t_2) \otimes W_2 \right) = 0. \tag{3.30}$$

Therefore,

$$H^s \left(\mathbb{P}^1, R^1 \beta_{2*}(R^1 \pi_{2*} \mathcal{F}) \right) = 0, \qquad s = 0, 1 \tag{3.31}$$

and the Leray tableau eq. (3.29) degenerates to

$$
\begin{array}{c|c|c|}
t=1 & 0 & 0 \\
\hline
t=0 & H^0 \left(\mathbb{P}^1, \beta_{2*}(R^1 \pi_{2*} \mathcal{F}) \right) & H^0 \left(\mathbb{P}^1, \beta_{2*}(R^1 \pi_{2*} \mathcal{F}) \right) \\
\hline
& s=0 & s=1
\end{array}
\tag{3.32}
$$

One must now compute the two cohomologies in the lower row, corresponding to $t = 0$. It was shown in [22] that

$$\beta_{2*} \left(\mathcal{O}_{B_2}(2t_2) \otimes W_2 \right) = \mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}. \tag{3.33}$$

Then from eq. (3.28) one finds that

$$\beta_{2*} \left(R^1 \pi_{2*} \mathcal{F} \right) = \left[\mathcal{O}_{\mathbb{P}^1}(-3)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1} \right]^{\oplus 36}. \tag{3.34}$$

Clearly, then

$$h^0 \left(\mathbb{P}^1, \beta_{2*}(R^1 \pi_{2*} \mathcal{F}) \right) = 36. \tag{3.35}$$

Using results from [22], we can obtain the corresponding 36-dimensional $\mathbb{Z}_3 \times \mathbb{Z}_3$ representation, and conclude that

$$H^0 \left(\mathbb{P}^1, \beta_{2*}(R^1 \pi_{2*} \mathcal{F}) \right) = RG^{\oplus 4}, \tag{3.36}$$

where RG stands for the nine-dimensional "regular representation" of $\mathbb{Z}_3 \times \mathbb{Z}_3$ given by

$$
\begin{aligned}
RG &= \bigoplus_{0 \le n, m \le 2} \chi_1^n \chi_2^m \\
&= 1 \oplus \chi_1 \oplus \chi_2 \oplus \chi_1^2 \oplus \chi_2^2 \oplus \chi_1 \chi_2 \oplus \chi_1 \chi_2^2 \oplus \chi_1^2 \chi_2 \oplus \chi_1^2 \chi_2^2.
\end{aligned}
\tag{3.37}
$$

Applying Serre duality on $\mathbb{P}^1$, and using the fact that the canonical bundle of $\mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}(-2)$, it follows from eq. (3.36) that

$$H^1\left(\mathbb{P}^1, \beta_{2*}(R^1\pi_{2*}\mathcal{F})\right) = RG^{\oplus 16}. \tag{3.38}$$

These results fill out the remaining entries in the Leray tableau eq. (3.29) for the push-down onto $\mathbb{P}^1$. The complete tableau is

$$
\begin{array}{c|c|c|}
t=1 & 0 & 0 \\
\hline
t=0 & RG^{\oplus 4} & RG^{\oplus 16} \\
\hline
& s=0 & s=1
\end{array}
\tag{3.39}
$$

Summing the diagonals in eq. (3.39), we can finally evaluate the $q = 1$ cohomologies $H^p(B_2, R^1\pi_{2*}\mathcal{F})$ in the first Leray spectral sequence. Recall, that $p = 0, 1, 2$ and that $s + t = p$. Then

1. $p = 0 \Rightarrow s = t = 0$:
$$H^0\left(B_2, R^1\pi_{2*}\mathcal{F}\right) = RG^{\oplus 4}, \tag{3.40}$$

2. $p = 1 \Rightarrow (s = 0, t = 1)$ or $(s = 1, t = 0)$:
$$H^1\left(B_2, R^1\pi_{2*}\mathcal{F}\right) = RG^{\oplus 16}, \tag{3.41}$$

3. $p = 2 \Rightarrow s = t = 1$:
$$H^2\left(B_2, R^1\pi_{2*}\mathcal{F}\right) = 0. \tag{3.42}$$

Therefore the complete Leray tableau eq. (3.20) for the push-down from $\widetilde{X}$ to B_2 is

$$
\begin{array}{c|c|c|c|}
q=1 & RG^{\oplus 4} & RG^{\oplus 16} & 0 \\
\hline
q=0 & 0 & 0 & 0 \\
\hline
& p=0 & p=1 & p=2
\end{array}
\tag{3.43}
$$

With this information one can, at last, compute the cohomologies B given in eq. (3.12). To do this, use the entries in eqns. (3.43) and (3.14), recalling that $m = p + q$. The results are

$$
\begin{aligned}
H^0\left(\widetilde{X}, V_1 \otimes V_2^\vee\right) &= 0, & H^1\left(\widetilde{X}, V_1 \otimes V_2^\vee\right) &= RG^{\oplus 4}, \\
H^2\left(\widetilde{X}, V_1 \otimes V_2^\vee\right) &= RG^{\oplus 16}, & H^3\left(\widetilde{X}, V_1 \otimes V_2^\vee\right) &= 0.
\end{aligned}
\tag{3.44}
$$

Cohomologies C. Cohomologies C can be computed directly from the cohomologies B in eq. (3.44). To do this, one uses Serre duality, the fact that, since $\widetilde{X}$ is a Calabi-Yau manifold, its canonical bundle is $\mathcal{O}_{\widetilde{X}}$ and the property that RG, given in eq. (3.37), is self-dual. It follows that the Leray tableau for the push-down from $\widetilde{X}$ to B_2 is

$$
\begin{array}{c|c|c|c|}
q=1 & 0 & 0 & 0 \\
\hline
q=0 & 0 & RG^{\oplus 16} & RG^{\oplus 4} \\
\hline
& p=0 & p=1 & p=2
\end{array}
\tag{3.45}
$$

and, therefore,

$$H^0\left(\widetilde{X}, V_2 \otimes V_1^\vee\right) = 0\,, \qquad H^1\left(\widetilde{X}, V_2 \otimes V_1^\vee\right) = RG^{\oplus 16}\,,$$
$$H^2\left(\widetilde{X}, V_2 \otimes V_1^\vee\right) = RG^{\oplus 4}\,, \qquad H^3\left(\widetilde{X}, V_2 \otimes V_1^\vee\right) = 0\,. \tag{3.46}$$

Cohomologies D. Cohomologies D are evaluated in much the same way as the B cohomologies. However, the calculation is harder and rather unenlightening. For these reasons, we will only state the results. We find that the Leray tableau for the push-down from $\widetilde{X}$ to B_2 is

$q=1$	0	ρ_{33}	1
$q=0$	1	ρ_{33}	0
	$p=0$	$p=1$	$p=2$

$$\tag{3.47}$$

where ρ_{33} is a specific 33-dimensional representation of $\mathbb{Z}_3 \times \mathbb{Z}_3$ given by

$$\rho_{33} = RG^{\oplus 3} \oplus \chi_1 \oplus \chi_2 \oplus \chi_1^2 \oplus \chi_2^2 \oplus \chi_1^2\chi_2 \oplus \chi_1\chi_2^2\,. \tag{3.48}$$

Therefore,

$$H^0\left(\widetilde{X}, V_2 \otimes V_2^\vee\right) = 1\,, \qquad H^1\left(\widetilde{X}, V_2 \otimes V_2^\vee\right) = \rho_{33}\,,$$
$$H^2\left(\widetilde{X}, V_2 \otimes V_2^\vee\right) = \rho_{33}\,, \qquad H^3\left(\widetilde{X}, V_2 \otimes V_2^\vee\right) = 1\,. \tag{3.49}$$

4. The long exact sequences

We now systematically proceed to compute the remaining cohomology spaces eq. (3.5) that will be required to evaluate $H^1\left(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\right)$. An important formula that will be used over and over again in our analysis is the following. Consider an exact sequence

$$\ldots \longrightarrow \mathcal{U} \xrightarrow{f_1} \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow \mathcal{X} \xrightarrow{f_2} \mathcal{Y} \longrightarrow \ldots\,. \tag{4.1}$$

Then

$$\dim_\mathbb{C}(\mathcal{W}) = \dim_\mathbb{C}(\mathcal{V}) + \dim_\mathbb{C}(\mathcal{X}) - \mathrm{rank}(f_1) - \mathrm{rank}(f_2)\,. \tag{4.2}$$

4.1 The H^0 cohomologies

We first focus on the 3×3 block of H^0 cohomologies in eq. (3.5). Using the "corner

cohomologies" computed in the previous section, the block is

, (4.3)

where we have labeled coboundary maps d_1, d_2, and d_3. The bottom horizontal exact sequence of this box is

$$0 \longrightarrow 1 \longrightarrow H^0\Big(\widetilde{X}, \widetilde{V} \otimes V_2^\vee\Big) \longrightarrow 0 \longrightarrow \rho_{33} \,. \tag{4.4}$$

Using formula eq. (4.2), we find immediately that

$$H^0\Big(\widetilde{X}, \widetilde{V} \otimes V_2^\vee\Big) = 1 \,. \tag{4.5}$$

Similarly, the right hand vertical exact sequence is

$$0 \longrightarrow 4 \longrightarrow H^0\Big(\widetilde{X}, V_1 \otimes \widetilde{V}^\vee\Big) \longrightarrow 0 \longrightarrow 0 \,. \tag{4.6}$$

It then follows from eq. (4.2) that

$$H^0\Big(\widetilde{X}, V_1 \otimes \widetilde{V}^\vee\Big) = 4 \,. \tag{4.7}$$

It remains to determine $H^0\big(\widetilde{X}, \widetilde{V} \otimes V_1^\vee\big)$ and $H^0\big(\widetilde{X}, V_2 \otimes \widetilde{V}^\vee\big)$ to complete the H^0 block, eq. (4.3). To do that, we need to know the three coboundary maps d_1, d_2, and d_3. First, consider the top horizontal exact sequence

$$0 \longrightarrow 0 \longrightarrow H^0\Big(\widetilde{X}, \widetilde{V} \otimes V_1^\vee\Big) \longrightarrow 4 \xrightarrow{d_2} RG^{\oplus 16} \,. \tag{4.8}$$

To evaluate $H^0\big(\widetilde{X}, \widetilde{V} \otimes V_1^\vee\big)$, we note that

$$d_2 : H^0\Big(\widetilde{X}, \mathcal{O}_{\widetilde{X}}^{\oplus 4}\Big) \to H^1\Big(\widetilde{X}, V_2 \otimes V_1^\vee\Big) \tag{4.9}$$

is multiplication of constant sections by a choice of extension in $\mathrm{Ext}_{\widetilde{X}}^1\big(V_1, V_2\big)$. For a generic choice of extension, it follows that d_2 is an injective map. This then implies that $\ker(d_2) = 0$ and, hence, that $\mathrm{rank}(d_2) = 4$. Using this result, eqns. (4.2) and (4.8) give

$$H^0\Big(\widetilde{X}, \widetilde{V} \otimes V_1^\vee\Big) = 0 \,. \tag{4.10}$$

Next, consider the left hand vertical exact sequence

$$0 \longrightarrow 0 \longrightarrow H^0\left(\widetilde{X}, V_2 \otimes \widetilde{V}^\vee\right) \longrightarrow 1 \xrightarrow{d_1} RG^{\oplus 16}\,. \tag{4.11}$$

An identical proof implies that $\mathrm{rank}(d_1) = 1$ and, hence, using eq. (4.2) we find

$$H^0\left(\widetilde{X}, V_2 \otimes \widetilde{V}^\vee\right) = 0\,. \tag{4.12}$$

The last unknown H^0 cohomology, $H^0\left(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\right)$, is contained in the middle vertical exact sequence given by

$$0 \longrightarrow 0 \longrightarrow H^0\left(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\right) \longrightarrow 1 \xrightarrow{d_3} H^1\left(\widetilde{X}, \widetilde{V} \otimes V_1^\vee\right)\,. \tag{4.13}$$

It follows from eq. (4.2) that

$$H^0\left(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\right) = 1 - \mathrm{rank}(d_3) \cdot 1\,. \tag{4.14}$$

Note that $\mathrm{rank}(d_3)$ can be either 0 or 1. Were $\mathrm{rank}(d_3) = 1$, then one would conclude from eq. (4.14) that $H^0\left(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\right)$ vanishes. But this is impossible, because

$$H^0\left(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\right) = H^0\left(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\right) \oplus H^0\left(\widetilde{X}, (\widetilde{V} \otimes \widetilde{V}^\vee)_{\mathrm{traceless}}\right)\,. \tag{4.15}$$
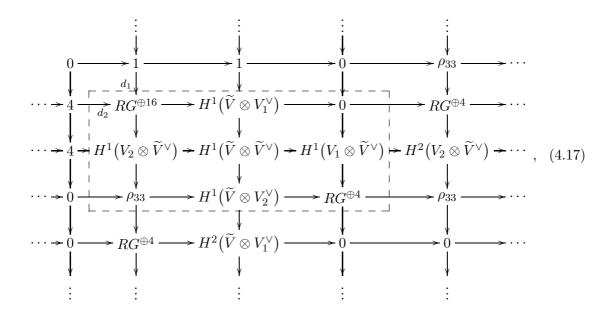
Then, using $H^0\left(\widetilde{X}, \mathcal{O}_{\widetilde{X}}\right) = 1$ we see that $h^0(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee) \geq 1$. Therefore, $\mathrm{rank}(d_3) = 0$ and eq. (4.14) implies

$$H^0\left(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\right) = 1\,. \tag{4.16}$$

In addition to completing the evaluation of the H^0 cohomologies, eq. (4.16) is important since it proves that the vector bundle $\widetilde{V}$ indeed satisfies the third non-trivial stability condition listed in eq. (2.12).

4.2 The H^1 cohomologies

We now focus on the 3×3 block of H^1 cohomologies in eq. (3.5). Since it contains the space of moduli, $H^1(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee)$, this is the final block that we need to consider. The H^1

cohomology block is

$$
\begin{array}{ccccccccc}
& \vdots & & \vdots & & \vdots & & \vdots & \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & 0 & \longrightarrow & \rho_{33} & \longrightarrow & \cdots \\
\downarrow & & {\scriptstyle d_1}\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to 4 & \to & RG^{\oplus 16} & \longrightarrow & H^1\big(\widetilde{V} \otimes V_1^\vee\big) & \longrightarrow & 0 & \longrightarrow & RG^{\oplus 4} & \longrightarrow & \cdots \\
\downarrow & {\scriptstyle d_2} & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to 4 & \to & H^1\big(V_2 \otimes \widetilde{V}^\vee\big) & \to & H^1\big(\widetilde{V} \otimes \widetilde{V}^\vee\big) & \to & H^1\big(V_1 \otimes \widetilde{V}^\vee\big) & \to & H^2\big(V_2 \otimes \widetilde{V}^\vee\big) & \to & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to 0 & \longrightarrow & \rho_{33} & \longrightarrow & H^1\big(\widetilde{V} \otimes V_2^\vee\big) & \longrightarrow & RG^{\oplus 4} & \longrightarrow & \rho_{33} & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to 0 & \longrightarrow & RG^{\oplus 4} & \longrightarrow & H^2\big(\widetilde{V} \otimes V_1^\vee\big) & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
& \vdots & & \vdots & & \vdots & & \vdots &
\end{array}
\quad , \quad (4.17)
$$

where we have inserted the "corner" cohomologies A, B, C and D as well as the H^0 results derived above. We immediately note that $H^1\big(\widetilde{X}, V_1 \otimes \widetilde{V}^\vee\big)$ lies in the right hand vertical sequence

$$
0 \longrightarrow 0 \longrightarrow H^1\Big(\widetilde{X}, V_1 \otimes \widetilde{V}^\vee\Big) \longrightarrow RG^{\oplus 4} \longrightarrow 0 \,. \tag{4.18}
$$

It then follows from eq. (4.2) that

$$
H^1\Big(\widetilde{X}, V_1 \otimes \widetilde{V}^\vee\Big) = RG^{\oplus 4} \,. \tag{4.19}
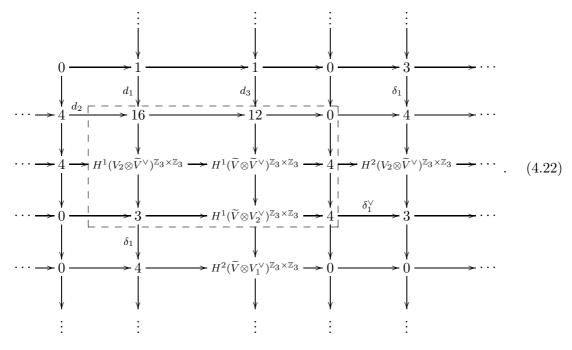$$

Similarly, one determines that

$$
H^1\Big(\widetilde{X}, \widetilde{V} \otimes V_1^\vee\Big) = RG^{\oplus 16} - \mathrm{rank}(d_2) \cdot 1 = RG^{\oplus 16} - 4 \,. \tag{4.20}
$$

We now proceed to evaluate the remaining elements in the H^1 block. To do this, it is essential that one knows the ranks of several coboundary maps in the intertwined sequences. These are hard to determine for the complete cohomology spaces. The problem simplifies, however, if we restrict the complex of sequences to the $\mathbb{Z}_3 \times \mathbb{Z}_3$ invariant subspace of each cohomology space. Then, using the fact that

$$
RG^{\mathbb{Z}_3 \times \mathbb{Z}_3} = 1 \qquad \rho_{33}^{\mathbb{Z}_3 \times \mathbb{Z}_3} = 3 \,, \tag{4.21}
$$

which follow from eqns. (3.37) and (3.48) respectively, the H^1 block and its nearby coho-

mologies simplify to

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 1 & \longrightarrow & 1 & \longrightarrow & 0 & \longrightarrow & 3 & \longrightarrow & \cdots \\
\downarrow & & \downarrow{\scriptstyle d_1} & & \downarrow{\scriptstyle d_3} & & \downarrow & & \downarrow{\scriptstyle \delta_1} \\
\cdots \rightarrow 4 & \xrightarrow{d_2} & 16 & \longrightarrow & 12 & \longrightarrow & 0 & \longrightarrow & 4 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \rightarrow 4 & \rightarrow & H^1(V_2\otimes\widetilde{V}^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3} & \rightarrow & H^1(\widetilde{V}\otimes\widetilde{V}^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3} & \rightarrow & 4 & \rightarrow & H^2(V_2\otimes\widetilde{V}^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3} & \rightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \delta_1^\vee} \\
\cdots \rightarrow 0 & \longrightarrow & 3 & \rightarrow & H^1(\widetilde{V}\otimes V_2^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3} & \rightarrow & 4 & \rightarrow & 3 & \rightarrow & \cdots \\
\downarrow & & \downarrow{\scriptstyle \delta_1} & & \downarrow & & \downarrow & & \downarrow \\
\cdots \rightarrow 0 & \longrightarrow & 4 & \rightarrow & H^2(\widetilde{V}\otimes V_1^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3} & \rightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
\tag{4.22}
$$

Note that we have indicated two new coboundary maps δ_1 and $\delta_1^\vee$ in eq. (4.22), as well as the maps d_1, d_2, and d_3 introduced previously.

For the invariant cohomology subspaces, one can show

$$
\delta_1 = 0 \tag{4.23}
$$

using the cup product in the Leray spectral sequence. We postpone the details to appendix. It is exactly at this point that we found it expedient to restrict to the invariant part of the cohomologies. Noting that $\delta_1^\vee$ is the Serre dual of δ_1, it follows that

$$
\delta_1^\vee = 0 \tag{4.24}
$$

as well. To compute the H^1 cohomologies, we must first know $H^2(\widetilde{X}, V_2 \otimes \widetilde{V}^\vee)^{\mathbb{Z}_3\times\mathbb{Z}_3}$. This lies in the vertical sequence

$$
3 \xrightarrow{\delta_1} 4 \longrightarrow H^2\left(\widetilde{X}, V_2 \otimes \widetilde{V}^\vee\right)^{\mathbb{Z}_3\times\mathbb{Z}_3} \longrightarrow 3 \longrightarrow 0 \,. \tag{4.25}
$$

Using eq. (4.23), we immediately obtain

$$
H^2\left(\widetilde{X}, V_2 \otimes \widetilde{V}^\vee\right)^{\mathbb{Z}_3\times\mathbb{Z}_3} = 7 \,. \tag{4.26}
$$

Serre duality then implies that

$$
H^1\left(\widetilde{X}, \widetilde{V} \otimes V_2^\vee\right)^{\mathbb{Z}_3\times\mathbb{Z}_3} = 7 \tag{4.27}
$$

as well. Note that this is consistent with the lower horizontal sequence in the H^1 block and eq. (4.24).

Let us now consider the left hand vertical long exact sequence in the H^1 block, which reads in part

$$1 \xrightarrow{d_1} 16 \longrightarrow H^1\left(\widetilde{X}, V_2 \otimes \widetilde{V}^\vee\right)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \longrightarrow 3 \xrightarrow{\delta_1} 4\,. \tag{4.28}$$
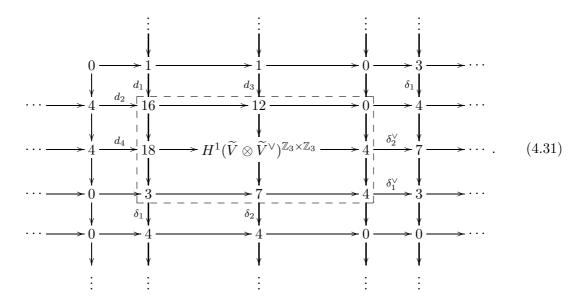
Using eq. (4.23), the fact, previously established, that $\mathrm{rank}(d_1) = 1$ and eq. (4.2), we find that

$$H^1\left(\widetilde{X}, V_2 \otimes \widetilde{V}^\vee\right)^{\mathbb{Z}_3 \times \mathbb{Z}_3} = 18\,. \tag{4.29}$$

Serre duality then implies

$$H^2\left(\widetilde{X}, \widetilde{V} \otimes V_2^\vee\right)^{\mathbb{Z}_3 \times \mathbb{Z}_3} = 18\,. \tag{4.30}$$

Putting this information back into the complex of sequences, we arrive, finally, at

$$\tag{4.31}$$

Note that we have introduced yet more coboundary maps: d_4, δ_2, and $\delta_2^\vee$ (the Serre dual of δ_2).

5. The moduli

One can now solve for the tangent space to the moduli space, $H^1(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee)^{\mathbb{Z}_3 \times \mathbb{Z}_3}$, of the observable sector. Of course, the complex dimension of the tangent space equals the number of moduli. To do this, consider the middle horizontal sequence in eq. (4.31) given by

$$0 \longrightarrow 1 \longrightarrow 4 \xrightarrow{d_4} 18 \longrightarrow H^1\left(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\right)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \longrightarrow 4 \xrightarrow{\delta_2^\vee} 7\,. \tag{5.1}$$

One must now determine the rank of the coboundary maps d_4 and $\delta_2^\vee$. Since we are restricted to the invariant cohomology subspaces, one can apply methods identical to those used in appendix to prove eq. (4.23). Again, one finds that

$$\delta_2^\vee = 0\,. \tag{5.2}$$

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The rank of d_4 can be determined by the exactness of the sequence eq. (5.1). The beginning of this sequence is

$$0 \xrightarrow{\phi_1} 1 \xrightarrow{\phi_2} 4 \xrightarrow{d_4} 18 \,, \tag{5.3}$$

where we named the first two maps ϕ_1 and ϕ_2. Exactness implies that $\mathrm{im}(\phi_1) = \ker(\phi_2)$ and, hence, that $\ker(\phi_2) = 0$. It follows that ϕ_2 is injective and that $\mathrm{im}(\phi_2) = \ker(d_4) = 1$. Therefore, $\mathrm{rank}(d_4)$ is the difference $4 - 1 = 3$. That is,

$$\mathrm{rank}(d_4) = 3 \,. \tag{5.4}$$

Then, using eqns. (5.2), (5.4), and (4.2), the exact sequence eq. (5.1) tells us that the number of moduli of the observable sector vector bundle $V = \widetilde{V}/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is

$$n_{\mathrm{observable}} = h^1\left(\widetilde{X}, \widetilde{V} \otimes \widetilde{V}^\vee\right)^{\mathbb{Z}_3 \times \mathbb{Z}_3} = 19 \,. \tag{5.5}$$

6. The hidden sector moduli

In the previous section, we computed the number of vector bundle moduli in the observable E_8 gauge sector. However, there is also the E_8' hidden sector (in the following, the prime will always denote hidden sector quantities), which potentially contributes moduli fields to the low energy effective action. These moduli interact only gravitationally with the fields of the standard model and, therefore, are not immediately relevant. Nevertheless, we would like to compute the hidden sector moduli in this section. The reason is twofold. First, the stability and dynamics of the hidden sector vector bundles is important for the discussion of supersymmetry breaking via E_8' fermion condensation and, potentially, for cosmology. The second reason is that the computation uses exactly the same formalism as for the observable sector bundles. It serves, therefore, as another, simpler, example of our method. For specificity, we will consider the hidden sector of the strongly coupled heterotic string only. Our formalism is easily applied to the weak coupling case as well.

Recall that in [22], for the case of strong string coupling, we chose the E_8' hidden sector gauge bundle to be an $\mathrm{SU}(2)$ instanton V' over the Calabi-Yau threefold X. As usual, we work with the $\mathbb{Z}_3 \times \mathbb{Z}_3$-equivariant bundle $\widetilde{V}'$ on the universal covering space $\widetilde{X}$. The bundle $\widetilde{V}'$ was explicitly defined by the extension

$$0 \longrightarrow V_2' \longrightarrow \widetilde{V}' \longrightarrow V_1' \longrightarrow 0 \,, \tag{6.1}$$

where V_1' and V_2' are the line bundles

$$V_2' = \mathcal{O}_{\widetilde{X}}(2\tau_1 + \tau_2 - \phi) \,, \qquad V_1' = \left(V_2'\right)^\vee = \mathcal{O}_{\widetilde{X}}(-2\tau_1 - \tau_2 + \phi) \,. \tag{6.2}$$

Analogous to eq. (3.4), we find that $\widetilde{V}' \otimes \widetilde{V}'^\vee$ lives in a 3×3 square of short exact

sequences

$$0 \to \overbrace{\mathcal{O}_{\widetilde{X}}(4\tau_1 + 2\tau_2 - 2\phi)} \to \widetilde{V}' \otimes V_1'^{\vee} \longrightarrow \overbrace{\mathcal{O}_{\widetilde{X}}} \longrightarrow 0$$

$$0 \longrightarrow V_2' \otimes \widetilde{V}'^{\vee} \longrightarrow \widetilde{V}' \otimes \widetilde{V}'^{\vee} \longrightarrow V_1' \otimes \widetilde{V}'^{\vee} \longrightarrow 0 \,. \tag{6.3}$$

$$0 \longrightarrow \overbrace{\mathcal{O}_{\widetilde{X}}} \longrightarrow \widetilde{V}' \otimes V_2'^{\vee} \to \overbrace{\mathcal{O}_{\widetilde{X}}(-4\tau_1 - 2\tau_2 + 2\phi)} \to 0$$

We already computed the "corner cohomologies" in [22]. Here, we simply quote the result that

$$H^p\left(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(4\tau_1 + 2\tau_2 - 2\phi)\right) = H^{3-p}\left(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-4\tau_1 - 2\tau_2 + 2\phi)\right) = \begin{cases} RG^{\oplus 6} & p = 1 \\ 0 & p \neq 1 \,. \end{cases} \tag{6.4}$$

Therefore, the H^0 cohomology block reads

$$\begin{array}{ccccccc}
0 \longrightarrow & 0 \longrightarrow & H^0(\widetilde{V}' \otimes V_1'^{\vee}) \longrightarrow & 1 \xrightarrow{d_2'} & RG^{\oplus 6} \longrightarrow & \cdots \\
0 \to & H^0(V_2' \otimes \widetilde{V}'^{\vee}) \to & H^0(\widetilde{V}' \otimes \widetilde{V}'^{\vee}) \to & H^0(V_1' \otimes \widetilde{V}'^{\vee}) \to & H^1(V_2' \otimes \widetilde{V}'^{\vee}) \to & \cdots \\
0 \longrightarrow & 1 \longrightarrow & H^0(\widetilde{V}' \otimes V_2'^{\vee}) \longrightarrow & 0 \longrightarrow & 0 \longrightarrow & \cdots \\
\cdots \longrightarrow & RG^{\oplus 6} \longrightarrow & H^1(\widetilde{V}' \otimes V_1'^{\vee}) \longrightarrow & 0 \longrightarrow & 0 \longrightarrow & \cdots
\end{array} \tag{6.5}$$

and, using exactly the same reasoning as in Subsection 4.1, we find that d_1' and d_2' are injective. Exactness of the sequence then implies that

$$H^0\left(\widetilde{X}, \widetilde{V}' \otimes V_1'^{\vee}\right) = H^0\left(\widetilde{X}, V_2' \otimes \widetilde{V}'^{\vee}\right) = 0 \,,$$
$$H^0\left(\widetilde{X}, V_1' \otimes \widetilde{V}'^{\vee}\right) = H^0\left(\widetilde{X}, \widetilde{V}' \otimes V_2'^{\vee}\right) = H^0\left(\widetilde{X}, \widetilde{V}' \otimes \widetilde{V}'^{\vee}\right) = 1 \,. \tag{6.6}$$

We proceed to the H^1 cohomology block, which now becomes

$$
\begin{array}{c}
\text{(6.7)}
\end{array}
$$

Since we already determined that d_1' and d_2' inject, and therefore

$$\mathrm{coker}(d_1') = \mathrm{coker}(d_2') = RG^{\oplus 6} - 1\,, \tag{6.8}$$

we can directly read off that

$$
\begin{aligned}
H^1\Big(\widetilde{X}, \widetilde{V}' \otimes V_1'^{\vee}\Big) &= H^1\Big(\widetilde{X}, V_2' \otimes \widetilde{V}'^{\vee}\Big) = RG^{\oplus 6} - 1\,, \\
H^1\Big(\widetilde{X}, V_1' \otimes \widetilde{V}'^{\vee}\Big) &= H^1\Big(\widetilde{X}, \widetilde{V}' \otimes V_2'^{\vee}\Big) = 0
\end{aligned} \tag{6.9}
$$

from the long exact sequences eq. (6.7). Finally, the middle horizontal long exact sequence

$$0 \longrightarrow 0 \longrightarrow 1 \overset{\sim}{\longrightarrow} 1 \overset{0}{\longrightarrow} RG^{\oplus 6} - 1 \overset{\sim}{\longrightarrow} H^1\big(\widetilde{X}, \widetilde{V}' \otimes \widetilde{V}'^{\vee}\big) \longrightarrow 0 \longrightarrow \cdots \tag{6.10}$$

yields

$$H^1\Big(\widetilde{X}, \widetilde{V}' \otimes \widetilde{V}'^{\vee}\Big) = RG^{\oplus 6} - 1\,. \tag{6.11}$$

Hence, the number of vector bundle moduli of $V' = \widetilde{V}'/(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is the invariant part of eq. (6.11). Using eq. (4.21)), we find that

$$n_{hidden} = h^1\Big(\widetilde{X}, \widetilde{V}' \otimes \widetilde{V}'^{\vee}\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} = 6 - 1 = 5\,. \tag{6.12}$$

We conclude that there are 5 vector bundle moduli in the hidden sector of the strongly coupled string.

Acknowledgments

A. The coboundary map δ_1

The purpose of this appendix is to determine the coboundary map

$$\delta_1 : H^1\Big(\widetilde{X}, V_2 \otimes V_2^\vee\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \to H^2\Big(\widetilde{X}, V_2 \otimes V_1^\vee\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \tag{A.1}$$

associated with the short exact sequence of equivariant vector bundles (see eq. (3.3))

$$0 \longrightarrow V_2 \otimes V_1^\vee \longrightarrow V_2 \otimes \widetilde{V}^\vee \longrightarrow V_2 \otimes V_2^\vee \longrightarrow 0 \,. \tag{A.2}$$

The choice of extension $V_2 \otimes \widetilde{V}^\vee$ is precisely the choice of an element x of the Ext-space

$$x \in \mathrm{Ext}^1_{\widetilde{X}}\left(V_2 \otimes V_2^\vee,\ V_2 \otimes V_1^\vee\right)^{\mathbb{Z}_3 \times \mathbb{Z}_3} = H^1\Big(\widetilde{X}, V_2 \otimes V_1^\vee\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \,. \tag{A.3}$$

Therefore, x must determine the coboundary map δ_1. One finds that it is the cup product, that is, the usual wedge product combined with a suitable contraction of vector bundle indices,

$$\delta_1 : H^1\Big(\widetilde{X}, V_2 \otimes V_2^\vee\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \to H^2\Big(\widetilde{X}, V_2 \otimes V_1^\vee\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \,,\ v \mapsto v \wedge x \,. \tag{A.4}$$

Note that the cohomology degree is additive. Since x is a degree 1 cohomology class, the image of δ_1 is indeed of degree 2. Nevertheless, we claim that the product map

$$\wedge : \underbrace{H^1\Big(\widetilde{X}, V_2 \otimes V_2^\vee\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3}}_{\ni\, v} \otimes \underbrace{H^1\Big(\widetilde{X}, V_2 \otimes V_1^\vee\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3}}_{\ni\, x} \to \underbrace{H^2\Big(\widetilde{X}, V_2 \otimes V_1^\vee\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3}}_{\ni\, \delta_1(v) = v \wedge x} \tag{A.5}$$

vanishes because of a refined degree stemming from the elliptic fibration. This can be seen as follows. Let us determine the cohomology spaces using the Leray spectral sequences eqns. (3.45) and (3.47) corresponding to the $\pi_2 : \widetilde{X} \to B_2$ fibration. First, note that the cohomology always comes from the π_{2*} push-down, and not the $R^1\pi_{2*}$ part:

$$
H^1\Big(\widetilde{X}, V_2 \otimes V_2^\vee\Big) = \rho_{33} \quad \Leftarrow
$$

	$p=0$	$p=1$	$p=2$
$q=1$	0	ρ_{33}	1
$q=0$	1	$\boldsymbol{\rho_{33}}$	0

$$
H^1\Big(\widetilde{X}, V_2 \otimes V_1^\vee\Big) = RG^{\oplus 16} \quad \Leftarrow
$$

	$p=0$	$p=1$	$p=2$
$q=1$	0	0	0
$q=0$	0	$\boldsymbol{RG^{\oplus 16}}$	$RG^{\oplus 4}$

$$
H^2\Big(\widetilde{X}, V_2 \otimes V_1^\vee\Big) = RG^{\oplus 4} \quad \Leftarrow
$$

	$p=0$	$p=1$	$p=2$
$q=1$	0	0	0
$q=0$	0	$RG^{\oplus 16}$	$\boldsymbol{RG^{\oplus 4}}$

$$\tag{A.6}$$

where we marked the relevant entry in the corresponding tableau in **bold face**. Hence, the product map, eq. (A.5), simplifies to

$$\wedge : H^1\Big(\widetilde{X}, \pi_{2*}\big(V_2 \otimes V_2^\vee\big)\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \otimes H^1\Big(\widetilde{X}, \pi_{2*}\big(V_2 \otimes V_1^\vee\big)\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \to$$

$$\rightarrow H^2\Big(\widetilde{X}, \pi_{2*}\big(V_2 \otimes V_1^\vee\big)\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3}. \tag{A.7}$$

These cohomology spaces are, in turn, determined by the Leray spectral sequence corresponding to the $\beta_2 : B_2 \rightarrow \mathbb{P}^1$ fibration:

$$H^1\Big(\widetilde{X}, \pi_{2*}\big(V_2 \otimes V_2^\vee\big)\Big) = \rho_{33} \quad \Leftarrow \quad$$

	$s=0$	$s=1$
$t=1$	$\color{red}{RG^{\oplus 3}}$	0
$t=0$	1	$\color{red}{(\chi_1+\chi_1^2)(1+\chi_2+\chi_2^2)}$

$$H^1\Big(\widetilde{X}, \pi_{2*}\big(V_2 \otimes V_1^\vee\big)\Big) = RG^{\oplus 16} \quad \Leftarrow \quad$$

	$s=0$	$s=1$
$t=1$	$\color{red}{RG^{\oplus 16}}$	$RG^{\oplus 4}$
$t=0$	0	0

$$H^2\Big(\widetilde{X}, \pi_{2*}\big(V_2 \otimes V_1^\vee\big)\Big) = RG^{\oplus 4} \quad \Leftarrow \quad$$

	$s=0$	$s=1$
$t=1$	$RG^{\oplus 16}$	$\color{red}{RG^{\oplus 4}}$
$t=0$	0	0

$$\tag{A.8}$$

where we notice that only the $R^1\beta_{2*}$ push-down contributes to the invariant part of the cohomology spaces. Hence, the product map, eq. (A.7), simplifies once more to

$$\wedge : H^0\Big(\widetilde{X}, R^1\beta_{2*}\pi_{2*}\big(V_2 \otimes V_2^\vee\big)\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \otimes H^0\Big(\widetilde{X}, R^1\beta_{2*}\pi_{2*}\big(V_2 \otimes V_1^\vee\big)\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3} \rightarrow$$
$$\rightarrow H^1\Big(\widetilde{X}, R^1\beta_{2*}\pi_{2*}\big(V_2 \otimes V_1^\vee\big)\Big)^{\mathbb{Z}_3 \times \mathbb{Z}_3}. \tag{A.9}$$

But this product is now zero for degree reasons: the product of two degree 0 cohomology spaces is again of degree 0, and not 1. Therefore, the product map specified in eqns. (A.9), (A.7), and (A.5) is the zero map. That is,

$$\delta_1 = 0, \tag{A.10}$$

as claimed.

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